RELATÓRIO DO WORKSHOP SOBRE TEORIA DE VALORIZÇÕES

Coordenador do Evento: ANTONIO JOSÉ ENGLER
Instituição Hospedeiro do Evento: INSTITUTO DE MATEMÁTICA, ESTATÍSITICA E COMPUTAÇÃO CIENTÍFICA – UNICAMP – Campinas

No período de 20 a 24 de março de 2006 realizou-se nas dependências do Instituto de Matemática, Estatística e Computação Científica da UNICAMP a reunião internacional intitulada "Workshop on Valuation Theory and its Applications" que foi dedicada a memória do Professor Otto Endler.

Descrição e avaliação do congresso: Nossa avaliação é que o workshop foi bem sucedido tendo-se obtido a maioria dos resultados esperados. Tivemos uma participação de 17 pesquisadores do exterior, vindos de vários países e de centros de pesquisa em matemática de reconhecida qualidade científica. Do país tivemos a participação de 18 pesquisadores externos a Universidade e 5 pesquisadores da Instituição além de 12 alunos de doutorado em álgebra. Conforme avaliação oral feita por vários participantes, o nível das conferências foi muito bom. Podemos em particular destacar o trabalho do Prof. Thomas Scanlon de Berkely apresentando (pela primeira vez publicamente) a resolução de uma conjectura de Pop (Pop's Conjecture on Relative Categoricity of Finitely Generated Fields*)* que despertou um grande interesse entre os participantes. Houve um bom interrelacionamento entre os participantes. Pudemos constatar que os alunos de doutorado presentes beneficiaram-se com a experiência científica dos participantes externos a Instituição. Embora nenhum resultado original tenha sido obtido, ou encaminhado, durante o congresso, vários problemas de pesquisa na área de valorizações (e temas relacionados) foram examinados quanto ao interesse e profundidade. Temos fortes razões para crer que várias cooperações iniciadas durante o congresso terão continuidade. Particularmente o Prof. David Brink de Copenhagen está considerando a possibilidade de realizar um programa de pós-doutorado na UNICAMP.

Dentre os participantes do exterior, o Prof. Yuri Ershov de Novosibirsk aproveitou sua viagem ao Brasil para visitar por 3 semanas o IME/USP; o Prof. Wulf-Dieter Geyer de Erlangen permaneceu por aproximadamente um mês na UFMG, Belo Horizonte, e a Profa. Sudesh Kaur Khanduja da Punjab University visitou o IMECC/UNICAMP por um mês. Particularmente em relação a essa última visitante estamos, eu e ela, escrevendo um artigo que deverá ser submetido a publicação muito em breve. Dessa forma a realização do workshop teve uma contribuição à atividade de pesquisa no país que ultrapassou seu período de realização.

Segue a seguir uma **lista competa dos participantes**: Mírian Abdon (IMPA), Alberto de Azevedo (UnB), João Lucas M. Barbosa (UFC), Gervásio G. Bastos (UFC), Valmecir Bayer (UFES), Eberhard Becker (Dortmund), David Brink (Copenhagen), Ludwig Broecker (Muenster), Rosali Brusamarello (UEM), Maximo A. Dickmann (Paris), Ada M. de Souza Doering (Porto Alegre), Yuri Ershov (Novosibirsk), Marcelo Escudeiro (UEM), Arnaldo Garcia (IMPA), Wulf-Dieter Geyer (Erlangen), Miguel Ferrero (UFRGS), Danielle Gondard (Paris), Abramo Hefez (UFF), Moshe Jarden (Tel Aviv), Sudesh K. Khanduja (Chandigarh, India), Manfred Knebusch (Regensburg), Jochen Koenignsmann (Princeton), Franz-Viktor Kuhlmann (Saskatoon, Canada), Yves Lequain (IMPA), Daniel Levcovitz (USP/São Carlos), Franciso Mirqaglia (USP), José G. de Oliveira (UFES), Alexander Prestel, (Konstanz), Cydara Ripoll (UFRGS), Osvaldo G. do Roccio (UEM), Thomas Scanlon (Berkeley), Aron Simis (UFP), Jean-Pierre Tignol (Louvain-La-Neuve, Belgian), Marcus Tressl (Regensburg).

Alunos Envolvidos: Sergio Mota Alves, Antonio P. Brandão Jr., Alonso S. Castellanos, Ronie P. Dario, Maurício de Araújo Ferreira, Cristiane A. Lázaro, Ângelo Papa Neto, Fabio A. de Matos, Paulo Cesar C. Oliveira, Claudenir F. Rodrigues, Ednei Aparecido Santulo Jr., Flávia S. M. da Silva, Nolmar Melo de Souza, Guilherme C. Tizziotti, Juan Elmer V. Zevallos.

Relação de trabalhos apresentados:

E. Becker (Dortmund, Germany)

Number of connected components of real projective varieties

Becker/Gondard proposed a formula for the number of connected components of a smooth real projective variety via the order of certain group of sums of powers. To derive it the authors made use of the space of real places, i.e. real valuations. Alternatively, there is a recent approach of Coliot-Thelene, based on a deep theorem of Rost and algebraic geometry ideas. The talk outlines both approaches and tries to speculate on the interrelationship between both ways.

D. Brink (Copenhagen, Denmark)

New light on Hensel's lemma.

Since the works of Kurt Hensel in the early twentieth century, numerous "Hensel's lemmas" have appeared, generalising and supplementing the original. In my talk, a new general Hensel's lemma will be given in which the discriminant and the resultant are replaced by two new polynomial invariants, the "separant" and the "bipartitionant". Examples suggest that this formulation is, in a certain sense, best possible. The proof uses a general Krasner's lemma as well as an accurate study of the continuity of roots and factors of polynomials.

L. Broecker (Muenster, Germany)

Reduction of semialgebraic constructible functions.

The image of a real algebraic variety under a polynomial map is no longer algebraic but merely semialgebraic. However, collecting Euler characteristics of the fibers one gets an algebraic function as image, that is a $\mathbb{Z}$-valued function which can be described as the sign-function of a quadratic form. We will show, that a corresponding result holds for the reduction of an algebraic variety with respect of a real valuation of the base field. Geometrically this means, that suitable limits of sequences of algebraic varieties in a family are actually algebraic functions, which encode multiplicities of the collapse.

M. Dickmann (Paris, France) and F. Miraglia (USP, São Paulo)

Generalized local-global principles, orders and real valuations.

This talk will deal with some results and open problems from the joint paper "Lattice-ordered reduced special groups" by M. Marshall, F. Miraglia and myself [Annal Pure Appl. Logic, **132** (2005),27-49], as well as with more recent developments. The local-global principles of the title are statements of a logical nature, which make sense in the context of special groups, and are motivated by —and, in fact, generalize— well-known results in quadratic form theory such as Pfister's local-global principle for isometry and Marashall's local-global principle for isotropy (the latter extending, in turn, older results from the case of fields to the context of abstract spaces of orders). The interest of such principles stems from the fact the their validity, even form restrict classes of logical formulas, entails interesting mathematical consequences. Results in our paper show that there is no hope to prove local-global principles which apply to properties significantly beyond those expresses by positive-existential formulas of the language of special groups (these already give ample leeway). Marshall has extended the proof of his isotropy theorems to show that the generalized local-global principle holds in all reduced special special groups for certain, combinatorially defined classes of (positive-existential) formulas. As for arbitrary formulas of this type, the generalized local-global principle holds in some cases —for example, the function field $\mathbb{Q}(X)$ as proved in the above mentioned paper— and fail in some others —for example, the function field of the "irrational circle" $x^2 + y^2 = 3$, as recently shown by Marshall; in many cases of geometrical interest —such as the field $\mathbb{R}(X,Y)$— its unrestricted validity is an open question.

We shall also deal with the case of formally real fields, in which a mathematically relevant transfer principle formulated in terms of orders and valuations follows from the generalized local-global principle.

Y. Ershov (Novosibirsk, Russia)

Lubin-Tate extentions (an elementary approach).

M. Escudeiro (UEM, Maringá)

An Effective Algorithm for the Analytic Classification of Plane Branches.

In this talk we present an effective algorithm for the classification of irreducible plane curve singularities. The main ingredient is a method that allow us to compute order of Kähler differentials using the normalizes valuation of the integral closure of the local ring of curve.

M. Ferrero (UFRGS, Porto Alegre)

On right chain semigroups

Right chain semigroups are semigroups in which right ideals are linearly ordered by inclusion. Multiplicative semigroups of right chain rings, right cones, right invariant right holoids and right valuation semigroups are examples. The ideal theory of right chain semigroups is described in terms of prime and completely prime ideals, and a classification of prime segments is given.

This lecture is based on a paper which is appearing in J. Algebra written in collaboration with Ryszard Mazurek and Alveri Sant'Ana.

A. Garcia (IMPA, Rio de Janeiro) Joint work with H. Stichtenoth (Univ.Duisburg-Essen)

Explicity towers over finite fields

This talk will cover some aspects of the theory of towers of curves (or of function fields) over finite fields. After the seminal work of A. Weil proving in particular an upper bound for the number of solutions with coordinates in a finite field (the so-called Hasse-Weil bound), which is equivalent to the validity of the Riemann Hypothesis for the associated Zeta Function, it took around thirty years till Ihara recognized that Weil's bound was weak for curves of high genus with respect to the cardinality of the finite field. Ihara then started the study of the asymptotics of such curves; i.e., fixing the finite field and letting the genus of the curves tend to infinity. Then came the work of Tsfasman-Vladut-Zink relating the asymptotics of curves with that of linear codes, leading to an improvement of the so-called Gilbert-Varshamov bound on the asymptotic transmission rates for codes. For practical applications one would like to have explicit equations for the curves or the function fields in the towers, and this is then the subject of this talk. We will also consider the Galois closures of some Artin-Schreier towers; i.e., towers where each step consists of an Artin-Schreier extension. We will derive new proofs of the asymptotic behavior of certain towers via the use of a very simple lemma on the compositum field of two Artin-Schreier extensions.

W.-D. Geyer (Erlangen, Germany) Joint work with (M. Jarden, Tel Aviv)

Non-PAC fields whose Henselian closures are separably closed.

A field K is called pseudo-algebraically closed (PAC) if every absolutely irreducible affine variety over it has a K-rational point. A PAC field is necessarily infinite, has no orderings, and all its nontrivial valuations have separably closed Henselizations. It had been a long-standing open problem whether every perfect field with these properties is necessarily PAC.
A related question is the following: Let F be an extension of relative transcendence degree 1 of a perfect field K. One says that the Hasse principle for Brauer groups holds for F/K if the restriction map $\mathrm{Br}(F) \to \prod_{v\in P(F/K)}\mathrm{Br}(F_v)$ of the Brauer groups is injective, where F_v runs over all Henselizations with respect to the nontrivial K-valuations v on F. I. Efrat (2001) proved that if K is any perfect field with the property that the Hasse principle holds for all extension fields of transcendence degree 1,

then all Henselizations of K with respect to nontrivial valuations having residue characteristic 0 are algebraically closed (a property of perfect PAC fields). Come out therefore the question whether every infinite perfect non-ordered field K such that the Hasse principle holds for every extension F of K of transcendence degree 1 would be necessarily PAC.

The following construction solves simultaneously both questions negatively. One starts with a finitely generated extension K_0 of $\mathbb{Q}$, and iteratively adjoins generic points of all absolutely irreducible varieties which are birationally equivalent over the algebraic closure to either a rational variety or an abelian variety. The resulting field K has no orderings, all its nontrivial valuations have algebraically closed Henselizations, and the Hasse principle holds for all extensions F of K of transcendence degree 1. Yet, curves of genus at least 2 over K_0 have no K-rational points, so K is not PAC.

D. Gondard-Cozette (Paris, France)

On Valuations Fans

We shall work in the frame of real fields, start with the compatibility of a preordering with a valuation, and then turn to the special case of fans and valuation fans. Afterward we shall present the theories of algebraic closures of a field equipped with a valuation fan.

A. Hefez (UFF, Rio de Janeiro)

The analytic classification of irreducible plane curve singularities

In this talk we show how to solve effectively the problem of analytic classification of plane branches proposed by O. Zariski in the 70's.

M. Jarden (Tel Aviv, Israel)

On the absolute Galois group of the maximal absolutely p-adic extension of $\mathbb{Q}$

We address on of the major problems of Galois theory: the characterization of absolute Galois groups among all profinite groups. Specifically, we consider a profinite group G equipped with a subset $\mathcal{G}$ each of it is isomorphic to an absolute Galois group. The problem is to characterize those pairs for which G is isomorphic to an absolute Galois group Gal(K) of a field K that satisfies a local-global principal for points on smooth varieties with respect to the fixed fields of the groups in $\mathcal{G}$. Let now $\mathcal{F}$ be the set of classical local fields of characteristic O. Thus, each $\mathsf{F}\in \mathcal{F}$ is either the field $\mathbb{R}$ of real numbers or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p. Put $\mathcal{C} = \{$ Gal$(\mathsf{F}) \mid \mathsf{F}\in \mathcal{F} \}$. Let $\mathcal{G}$ = Subgrt$(G,\mathcal{C})$be the set of all $H \in$ Subgrt(G) = the space of all closed subgroups of G (Subgrt(G) is the projective limit of the discrete finite spaces Subgrt(G/N) where N ranges over the set of all open normal subgroups of G. Thus Subgrt(G) is a profinite space.) which are isomorphic to some sugroup in $\mathcal{C}$. If a subset $\mathcal{X}$ of Subgrt(G) contains grupos A and B with $A < B$, then $\mathcal{X}$ is not étale Hausdorff. Thus, removing all non-maximal elements from $\mathcal{G}$ is the only way to make $\mathcal{G}$ étale Hausdorff while preserving the essential information stored in $\mathcal{G}$. We denote the set of all maximal elements of $\mathcal{G}$ by $\mathcal{G}_{max}$.

Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 and let K be a field. For each $\mathsf{F}\in\mathcal{F}$ let AlgExt(K,F) be the set of all algebraic extensions of K which are elementary equivalent to F. Put AlgExt$(K,\mathcal{F}) = \bigcup_{\mathsf{F}\in \mathcal{F}}$ AlgeExt(K,F). Call a field K **pseudo-$\mathcal{F}$-closed** (abbreviation P$\mathcal{F}$C) if every smooth absolutely irreducible variety over K, with an F- rational point for each $\mathsf{F}\in \mathcal{F}$, has a K-rational point. Finally, we call a profinite group G **strongly – $\mathcal{F}$ - projective,** if G is $\mathcal{G} = \bigcup_{\mathsf{F}\in \mathcal{F}}\{ H \in$ Subgrt$(G) \mid H \cong$ Gal$(\mathsf{F})\}$.

In order to formulate our main result we have the impose a restriction on the set $\mathcal{F}$: we say that $\mathcal{F}$ is **closed under Galois isomorphism** if for all classical local fields F, F' if $\mathsf{F}\in\mathcal{F}$ and Gal$(\mathsf{F}) \cong$ Gal(F'), then $\mathsf{F}'\in \mathcal{F}$.

Theorem: Let $\mathcal{F}$ be a finite set of classical local fields of characteristic 0 which is closed under Galois isomorphism. Let K be a P$\mathcal{F}$C field. Put

$$\mathcal{G} = \bigcup_{F\in\mathcal{F}} \{ \mathrm{Gal}(F) \mid F\in \mathrm{AlgExt}(K) \text{ and } \mathrm{Gal}(K) \cong \mathrm{Gal}(F) \}.$$

Then Gal(K) is strongly $\mathcal{F}$ – projective and $(\mathrm{Gal}(K), \mathcal{G}_{max})$ is a proper projective group structure.

Problem: Is it possible to remove the condition "$\mathcal{F}$ is closed under Galois isomorphism" from the Theorem?

<u>S. K. Khanduja</u> (Chandigarh, India)

Properties of Tignol's constant and its relation with different in valued field extensions

Let v be a Henselian valuation of arbitrary rank defined on a field K. Let (L,w) be a finite separable extension of (K,v) and $\mathrm{Tr}_{L/K}$ stand for the trace map of this extension. In 2004, it was proved that the set $A_{L/K} = \{v(\mathrm{Tr}_{L/K}(\alpha)) - w(\alpha) | \alpha\in L, \alpha \neq 0\}$ has a minimum element if and only if $(L,w)/(K,v)$ is a defectless extension. The constant **min**$A_{L/K}$ was first introduced by Tignol and it is referred to as Tignol's constant. In the lecture, properties of Tignol's constant, its relation with the different of $(L,w)/(K,v)$ and an explicit formula for this constant will be given which yield some well known results of Dedekind regarding ramification of prime ideals in local fields.

<u>M. Knebusch</u> (Regensburg, Germany) The lecture is a report on recent work with Digen Zhang.

Special valuations and Kronecker extensions

Any valuation $v{:}R \to \Gamma \cup\{\infty\}$ on a commutative ring R admits a unique maximal primary specialization v^*, called a *"special valuation"*. Thus special valuations abound in commutative ring theory. One would like to describe important classes of ring extensions $A \subset R$ in terms of families of special valuations. {A Krull domain A with R the quotient field of A is a classical case in point.}

This program works very well in the case of Prüfer extensions, where the relevant special valuations are Manis valuations, cf. our book, Springer LNM 1791. Now we are able to associate with A\subset R in a much broader class than Prüfer extensions a commuting square

$$\begin{array}{ccc} R & \hookrightarrow & T \\ \uparrow & & \uparrow \\ A & \hookrightarrow & B \end{array}$$

such that $B \subset T$ is Prüfer and reflects much of the ideal theory of $A \subset R$. The point is that the special valuations on R over A correspond uniquely with Manis valuations on T over B. The commuting square is gained by a construction which ultimately goes back to Kronecker.

<u>J. Koenignsmann</u> (Philadelphia, USA)

Anabelian geometry over almost arbitrary fields

I will give a survey on new results in anabelian geometry building prominently on techniques from valuation theory. I will prove that for almost all perfec fields K and for all function fields F in one variable over K, the absolute Galois group of F over K determines F and K up to isomorphism. In particular, almost all perfect fields K are up to isomorphism determined by the absolute Galois group of the rational function field $K(t)$ over K. This will be applied to give new axiomatizations of fields in Galois theoretic terms. Implications for the decidability of the local fields $\mathbb{F}_q((t))$ of positive characteristic will be given.

<u>F.-V. Kuhlmann</u> (Saskatoon, Canada)

The defect

If you want to find the maximum information on the defect in a textbook on Valuation Theory, you will have to look at Otto Endler's book. But since the time this book was written, a lot of work has been done to understand the defect. It has turned out that the defect is ultimately responsible for the problems we encounter when we want to prove local forms of resolution of singularities in positive characteristic, or want to prove that the elementary theory of formal Laurent series fields over finite fields is decidable. I will give a short introduction to these problems and show how the defect comes into the picture. Ramification theory tells us that in order to understand the defect, we have to consider first the Artin-Schreier extensions of valued fields (as well as the purely inseparable extensions). I will give an overview of recent work done on Artin-Schreier extensions, the classification of their defects and applications of this classification. I will give examples of extensions with non-trivial defect and a list of open problems.

A. Prestel (Konstanz, Germany)

On the development of valuation theory

Absolute values of a field and their completions - like the p-adic number fields - played an important role in the development of number theory in the beginning of the 20th century. In the 1930's Krull generalized the notion of an absolute value to that of a valuation. This generalization made possible applications in other branches of mathematics, such as algebraic and real algebraic geometry and more recently in the study of singularities. In the theory of valuations, the notion of a completion had to be replaced by that of the so-called henselization. The non-archimedean absolute values of the field $\mathbb{Q}$ of rational numbers are in one-to-one correspondence with the prime numbers p. Let v_p be the absolute value corresponding to p. The completion of $\mathbb{Q}$ with respect to v_p is known as the field $\mathbb{Q}_p$ of p-adic numbers. With its use we can formulate the famous and very useful local-global principle of Hasse-Minkowski concerning quadratic forms: Let $f(X_1,.....,X_n)$ be a homogeneous polynomial of degree 2 over $\mathbb{Q}$. Then f has a non-trivial zero in $\mathbb{Q}$ if it has one in $\mathbb{R}$ and one in each $\mathbb{Q}_p$. The reason this principle is so useful lies in the fact that solving equations is much easier in $\mathbb{R}$ and in $\mathbb{Q}_p$ than in $\mathbb{Q}$. For $\mathbb{R}$ this is pretty clear. This fact for $\mathbb{Q}_p$, or more generally for a field K that is complete with respect to a non-archimedean absolute value v, follows from “Hensel's Lemma” which sais that in solving polynomial equations one can reduce the problem to an equivalent problem on the finite field $\mathbb{F}_p$. For valuation of arbitrary rank, as a substitute for the completions, one therefore introduces a certain algebraic extension field K^h of K that canonically extends the valuation of K and satisfies Hensel's Lemma. This extension, which is unique up to value isomorphism, is called the henselization of (K, v). Although K need no longer be dense in K^h, the henselization still has the same residue class field and takes the same values as K, a property which, for the completion, is a consequence of the density. It is the henselization that opened up to mathematicians the opportunity for applications as mentioned above. In this talk we shall exploit the potencial of Hensel's Lemma in reducing degree of complexity of problems making it a far rich instrument for researchers.

T. Scanlon (Berkeley, USA)

Pop's Conjecture on Relative Categoricity of Finitely Generated Fields

We prove F. Pop's conjecture that two elementarily equivalent finitely fields must be isomorphic (at least in characteristic not equal to two) by showing that certain families of valuations are definable. More specifically, we show that if $K = k(C)$ is a function field of a curve over a finitely generated field k (of characteristic not equal to two) then the k-rational valuations on K are uniformly definable in the language of rings with parameters from K. As a consequence, we show that K is bi-interpretable with the ring of rational integers.

A. Simis (UFP, Recife)

Integral closure and the module of differentials.

In the last few years integral closure of modules has had a strong revival ever since the early days of Zariski's complete modules. We will briefly mention the basics of the theory and switch to some problems related to the module of differentials in this respect.

M. Tressl (Regensburg, Alemanha)

*Valuation theory of **o**-minimal structures*

O-minimal structures provide an axiomatic set-up for "tame" real analysis. Many functions, like the global real exponential function and many sets, like compact real analytic manifolds can be found in o-minimal structures. The tameness allows the application of algebraic methods, more particular valuation theoretic methods, in the geometric study of such sets. The analysis of semi-algebraic sets of real closed fields in terms of real algebraic geometry, serve as a model for the application of valuation theory in o-minimal structures. A good example is the Abhyankar-inequality, generalized to o-minimal structures, which translates into the existence of Hausdorff limits in o-minimal structures and the finiteness of characteristic exponents for global subanalytic sets. The techniques in the proofs for the o-minimal results are essentially different from the classical algebraic proofs as it is not known how o-minimal structures are built up from computational accessible objects, like polynomials. Instead, a substantial part of polynomial-like algebra and valuation theoretic statements can be formulated and proved in terms of Dedekind Cuts of o-minimal structures. I will give an overview of the matter from this point of view.